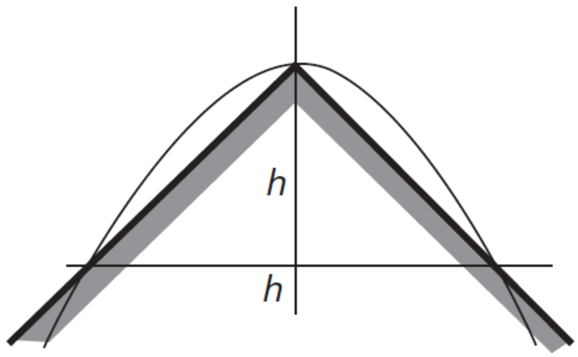
Problem 1.27

Peaked roof
A peaked roof is symmetrical and subtends a right angle, as shown. Standing at a height of distance h below the peak, with what initial speed must a ball be thrown so that it just clears the peak and hits the other side of the roof at the same height?



Solution

The trick to solving this problem is to split up the motion along the horizontal x and vertical y axes. A projectile that is fired with initial velocity v_0 at an angle θ has a component in the x-direction of $v_0 \cos\theta$ and a component in the y-direction of $v_0 \sin\theta$.

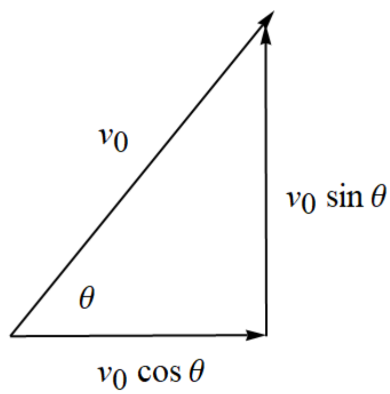


Figure 1: Decomposition of the initial velocity vector into components along the x and y axes.

The two kinematic equations that relate position, velocity, and acceleration are

$$v^2 - v_0^2 = 2a(x - x_0)$$
$$x = x_0 + v_0 t + \frac{1}{2}at^2.$$

These two equations will be applied along the x-axis and along the y-axis, giving us four equations in total. Choose the coordinate system in the following figure.

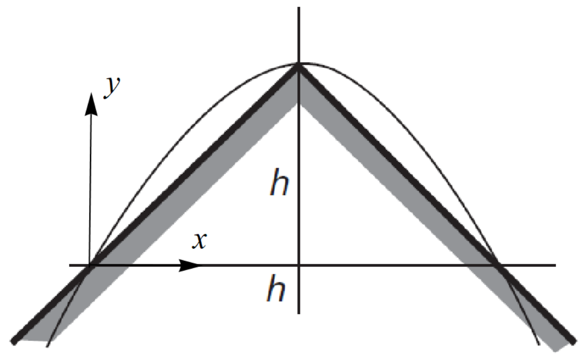


Figure 2: This is the chosen coordinate system for the peaked roof problem.

The initial position will be where the projectile fires from on the left $(x_0, y_0) = (0, 0)$, and the final position will be at the roof's peak $(x, y) = (h, h)$. Here we apply the two aforementioned equations in the x-direction. Note that the acceleration in this direction is zero, so the velocity at the roof's peak is the same as initially.

$$(v_0 \cos\theta)^2 - (v_0 \cos\theta)^2 = 2(0)h \quad \rightarrow \quad 0 = 0 \tag{1}$$

$$h = 0 + (v_0 \cos\theta)t + \frac{1}{2}(0)t^2 \quad \rightarrow \quad h = v_0(\cos\theta)t \tag{2}$$

Now apply the two aforementioned equations in the y-direction. The velocity falls to 0 once the projectile reaches the peak.

$$(0)^2 - (v_0 \sin\theta)^2 = 2(-g)h \quad \rightarrow \quad v_0^2 \sin^2\theta = 2gh \tag{3}$$

$$h = 0 + (v_0 \sin\theta)t + \frac{1}{2}(-g)t^2 \quad \rightarrow \quad h = v_0(\sin\theta)t - \frac{1}{2}gt^2 \tag{4}$$

We have three equations and three unknowns, v_0, θ, and t, so each of them can be solved for in terms of the known quantities, h and g. Work with equation (4) first.

$$h + \frac{1}{2}gt^2 = v_0(\sin\theta)t$$

Multiply both sides by 2.

$$2h + gt^2 = 2v_0(\sin\theta)t$$

Square both sides.

$$4h^2 + 4ght^2 + g^2t^4 = 4v_0^2(\sin^2\theta)t^2$$

Substitute equation (3) to the right side.

$$4h^2 + 4ght^2 + g^2t^4 = 4(2gh)t^2$$

$$4h^2 + 4ght^2 + g^2t^4 = 8ght^2$$

$$g^2t^4 - 4ght^2 + 4h^2 = 0$$

This is a quadratic equation for t^2. As such, use the quadratic formula.

$$\begin{aligned} t^2 &= \frac{4gh \pm \sqrt{\cancel{16g^2h^2} - \cancel{16g^2h^2}}}{2g^2} \\ &= \frac{2h}{g} \end{aligned}$$

So we have

$$t = \sqrt{\frac{2h}{g}}.$$

Substitute this result into equation (2).

$$h = v_0(\cos\theta)\sqrt{\frac{2h}{g}}$$

Solve this for $\cos\theta$.

$$\cos\theta = \frac{\sqrt{gh}}{v_0\sqrt{2}}$$

From this equation a right triangle can be constructed, and an expression for $\sin\theta$ can be deduced from it.

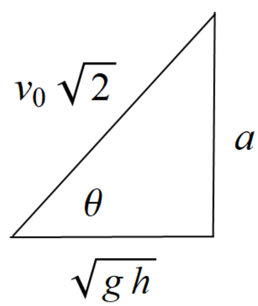


Figure 3: This right triangle is made from the expression for $\cos\theta$.

a can be determined with the Pythagorean theorem.

$$gh + a^2 = 2v_0^2 \quad \rightarrow \quad a = \sqrt{2v_0^2 - gh}$$

$\sin\theta$ is the ratio of the opposite side to the hypotenuse.

$$\sin\theta = \frac{\sqrt{2v_0^2 - gh}}{v_0\sqrt{2}}$$

Square both sides of this equation.

$$\sin^2\theta = \frac{2v_0^2 - gh}{2v_0^2}$$

Substitute this result into equation (3).

$$v_0^2\frac{2v_0^2 - gh}{2v_0^2} = 2gh$$

$$\frac{2v_0^2 - gh}{2} = 2gh$$
$$2v_0^2 - gh = 4gh$$
$$2v_0^2 = 5gh$$

Therefore,

$$v_0 = \sqrt{\frac{5}{2}}\sqrt{gh}.$$